Equidistant permutations by goober

(revision 2, 2012/02/05 – minor grammar fixes)

Problem source: `http://radoslav-harman.blogspot.com/2010/03/ekvidistantne-permutacie.html`

Problem setting: Let $n > 2$ be a natural number and σ be a permutation of the set $\{1, \ldots, n\}$. The number σ_k will be called k-th **term** of the permutation and σ_{k+1} and σ_{k-1} will be its **neighbours**; where the addition or subtraction of indices is considered as wrapping from n to 1 and back. Every permutation describes a sequence of n points in 2-dimensional Cartesian plane, with k-th point defined as $P_k = (\sigma_k, \sigma_{k+1})$. The permutation σ is **equidistant** if the Euclidean distances between points P_k and P_{k+1} are the same for all values of k (the usual wrap-around of indices applies to points P_k as well).

Rado asks: Which values of n admit an equidistant permutation of length n?

We'll start with a few simple observations about these beasts. First, let's see what happens if we put the distances $d(\cdot, \cdot)$ between three consecutive points into one equality:

$$
\begin{aligned}
d(P_k, P_{k+1}) &= d(P_{k+1}, P_{k+2}) \\
(\sigma_k - \sigma_{k+1})^2 + (\sigma_{k+1} - \sigma_{k+2})^2 &= (\sigma_{k+1} - \sigma_{k+2})^2 + (\sigma_{k+2} - \sigma_{k+3})^2 \\
(\sigma_k - \sigma_{k+1})^2 &= (\sigma_{k+2} - \sigma_{k+3})^2 \\
|\sigma_k - \sigma_{k+1}| &= |\sigma_{k+2} - \sigma_{k+3}|
\end{aligned}
$$

Thus, if we write down the absolute differences between consecutive terms of the permutation, we'll see the same value (A) on each even place, and another one every odd place (B). Clearly, these two values cannot be equal; otherwise the permutation would be forced to be an arithmetic progression[1] of length greater than two. However, the absolute difference of the first and last term would then be greater than the difference between consecutive terms; contradicting our assumption of equidistance. **Thus, each term of the permutation has one neighbour differing from it by A (its A-neighbour) and one differing by B (the B-neighbour).**

What can we say about A and B? If G denotes their greatest common divisor, all terms of the permutation fall into the same residue class modulo G. Since the permutation contains both numbers 1 and 2, G cannot be greater than 1. In other words, **the numbers A and B share no non-trivial common factor**.

Now, imagine that the pair (A, B) is given and we're trying find whether it corresponds to any equidistant permutation.
Which number will be the A-neighbour of 1? It could be either $(1 - A)$ or $(1 + A)$. The first one is smaller than 1 which is impossible, leaving only $(1 + A)$. The same reasoning can be applied to any number X between 1 and A (inclusive), forcing its A-neighbour to be $(A + X)$ (and $(A + X)$'s A-neighour to be X).
If $n > 2A$, we can do something similar with $(2A + 1)$ – the only candidates for its A-neighbour are $(A + 1)$ and $(3A + 1)$. However, $(A + 1)$ is already occupied as 1's neighbour; thus $(2A + 1)$ must be A-neighbour of $(3A + 1)$. In general, **for any non-negative integer K, the A-neighbour of the number $(2KA + X)$, where $1 \le X \le A$, is forced to be $(2KA + A + X)$ (and vice versa)**[2].
In fact, we can get even more from this observation – if $(2KA + 1)$ doesn't exceed n, its A-neighbour $(2KA + A + 1)$ cannot do that either. But then, neither does $(2KA + A)$ and neither can its A-neighbour $(2K + A + A) = 2(K + 1)A$. Thus, n must be a multiple of $2A$. The same

[1] In order to maintain the equidistance property without repeating any term twice.
[2] Naturally, we're assuming $(2KA + X) \le n$

type of reasoning can be applied to B, telling us that n is also divisible by $2B$. Since A and B share no common non-trivial factor, n must be divisible by $2AB$.
On the other hand, if we look at the numbers in the interval $1, \dots, 2AB$, the A-neighbour and B-neighbour of each of them falls into this interval as well. Thus, starting from 1, the permutation will never be able to reach any number greater than $2AB$, allowing us to conclude that n **is equal to** $2AB$.

Now, we're ready to look at the values of A and B which give rise to equidistant permutations. Without loss of generality, we can assume $1 \le A < B$. It'll be useful to write B in the form $B = TA + Q$, there $T \ge 1$ and $0 \le Q < A$ are integers.

The easiest case is when T is even. Consider the following figure:

$$
\begin{array}{|ccc|c|}
\hline
 & | & \downarrow_B & \\
0A+1 & = & 0B+1 & 1 \\
\downarrow_A & | & & \\
0A+(A+1) & = & 0B+(A+1) & 1+A \\
 & | & \downarrow_B & \\
TA+(A+Q+1) & = & 0B+(B+A+1) & 1+A+B \\
\downarrow_A & | & & \\
TA+(Q+1) & = & 0B+(B+1) & 1+B \\
 & | & \downarrow_B & \\
\hline
\end{array}
$$

If we start with number 1 and follow the chain of neighbours (starting with A-neighbour) according to the general rule described earlier, we'll necessarily follow the numbers listed in right column of the figure. The left-hand side shows the same numbers split into an even multiple of A and a positive remainder not exceeding $2A$ (or the same for B, to the right of the "equals" sign) along with arrows showing whether we followed A-neighbour or B-neighbour.
The last line shown in the table has to be followed by the first, forming a cycle. Since the numbers in a permutation cannot repeat, this configuration is only possible if $n = 4$. A trivial calculation then proves that **of all the pairs** (A, B) **with** T **even,** $(1, 2)$ **is the only one corresponding to an equidistant permutation** – $(1, 3, 4, 2)$.

Another simple case is when A is equal to 1 and B is odd. This gives rise to a permutation of the form

$$
\begin{array}{|ccc|}
\hline
 & & \searrow_A \\
1 & \longleftarrow_B & (1+B) \\
\downarrow_A & & \\
2 & \longrightarrow_B & (2+B) \\
 & & \downarrow_A \\
3 & \longleftarrow_B & (3+B) \\
\downarrow_A & & \\
 & \vdots & \\
 & & \downarrow_A \\
B & \longleftarrow_B & (B+B) \\
 & \searrow_A & \\
\hline
\end{array}
$$

This is a valid permutation of length $n = 2B$, so **an equidistant permutation exists for every** n **of the form** $n = 4k + 2$**, where** $k \ge 1$ **is an integer**[3].

Now that we have exhausted all values of B in combination with $A = 1$, we can assume that $A > 1$ and, thanks to the greatest-common-divisor condition, $0 < Q < A$ and write A in the form $A = 2MQ + P$, with $M \ge 0$ and $0 \le P < 2Q$ being integers.

[3]Technically, we could include the trivial case of $n = 2$ into this group as well.

We'll be done with the case $P \neq 0$ quickly, as shown in the following table whose form we're already familiar with. In order to simplify things a bit in the rightmost column, S will denote the number $S = 2BM + \lceil \frac{P}{2} \rceil$.

	$\vert$	$\downarrow_B$	
$(2MT)A + (A - \lfloor \frac{P}{2} \rfloor)$	$=$	$(2M)B + \lceil \frac{P}{2} \rceil$	S
$\downarrow_A$	$\vert$		
$(2MT)A + (2A - \lfloor \frac{P}{2} \rfloor)$	$=$	$(2M)B + (A + \lceil \frac{P}{2} \rceil)$	$S + A$
	$\vert$	$\downarrow_B$	
$(2MT + T + 1)A + (A + Q - \lfloor \frac{P}{2} \rfloor)$	$=$	$(2M)B + (B + A + \lceil \frac{P}{2} \rceil)$	$S + A + B$
$\downarrow_A$	$\vert$		
$(2MT + T + 1)A + (Q - \lfloor \frac{P}{2} \rfloor)$	$=$	$(2M)B + (B + \lceil \frac{P}{2} \rceil)$	$S + B$
	$\vert$	$\downarrow_B$	

The equality in the first row is based on $B = TA + Q$ and $A - \lfloor \frac{P}{2} \rfloor - \lceil \frac{P}{2} \rceil = A - P = 2MQ$, rest of the calculation should be pretty straight-forward. Once again, we've found a cycle consisting of four elements, which can exist only for $n = 4$. However, we have $B > A \geq 2$ now and S cannot be smaller than 1, so their sum cannot be smaller $(1 + 2 + 3) = 6$ – which is more than n. Thus, **there are no equidistant permutations with T odd and $P \neq 0$.**

In case of $P = 0$, we can write $A = 2MQ$ and $B = TA + Q = (2MT + 1)Q$. The greatest common divisor of A and B is now Q, so based on our previous observation, Q must be equal to 1, allowing us to write $A = 2M$, $B = TA + 1 = 2MT + 1$.
If $T > 1$, the following table says it all:

	$\vert$	$\downarrow_B$	
$0A + A$	$=$	$0B + A$	A
$\downarrow_A$	$\vert$		
$0A + 2A$	$=$	$0B + 2A$	$2A$
	$\vert$	$\downarrow_B$	
$(T + 1)A + (A + 1)$	$=$	$0B + (B + 2A)$	$2A + B$
$\downarrow_A$	$\vert$		
$(T + 1)A + 1$	$=$	$0B + (B + A)$	$A + B$
	$\vert$	$\downarrow_B$	

Once again, this could happen only if $n = 4$ and since A is the smallest of the numbers, it'd have to be equal to 1 – which is not allowed by our assumptions. Therefore, $T > 1$ **odd and** $P = 0$ **doesn't produce an equidistant permutation**.

The only remaining case is $T = 1$ and $P = 0$, when $(A, B) = (2M, 2M + 1)$ for some positive integer M. Then, the last and longest list tells us what happens:

	$\vert$	$\downarrow_B$	
$(2M-2)A+(A-1)$	$=$	$(2M-2)B+1$	A^2-A-1
$\downarrow_A$	$\vert$		
$(2M-2)A+(2A-1)$	$=$	$(2M-2)B+B$	A^2-1
	$\vert$	$\downarrow_B$	
$(2M)A+A$	$=$	$(2M-2)B+2B$	A^2+A
$\downarrow_A$	$\vert$		
$(2M)A+2A$	$=$	$(2M)B+A$	A^2+2A
	$\vert$	$\downarrow_B$	
$(2M+2)A+(A+1)$	$=$	$(2M)B+(A+B)$	A^2+3A+1
$\downarrow_A$	$\vert$		
$(2M+2)A+1$	$=$	$(2M)B+B$	A^2+2A+1
	$\vert$	$\downarrow_B$	
$(2M+2)A+(A+2)$	$=$	$(2M)B+2B$	A^2+3A+2
$\downarrow_A$	$\vert$		
$(2M+2)A+2$	$=$	$(2M)B+(B+1)$	A^2+2A+2
	$\vert$	$\downarrow_B$	
$(2M)A+(A+1)$	$=$	$(2M)B+1$	A^2+A+1
$\downarrow_A$	$\vert$		
$(2M)A+1$	$=$	$(2M-2)B+(B+2)$	A^2+1
	$\vert$	$\downarrow_B$	
$(2M-2)A+A$	$=$	$(2M-2)B+2$	A^2-A
$\downarrow_A$	$\vert$		
$(2M-2)A+2A$	$=$	$(2M-2)B+(B+1)$	A^2
	$\vert$	$\downarrow_B$	

These twelve numbers form a cycle, thus the corresponding equidistant permutation (if there is any) has $n = 12$. Setting the smallest of the expressions, $(A^2 - A - 1)$, equal to 1 yields $A = 2$ and $B = 3$. In other words, **of the pairs (A, B) with $T = 1$ and $P = 0$, only the pair $(2, 3)$ produces any equidistant permutation, namely $(1, 3, 6, 8, 11, 9, 12, 10, 7, 5, 2, 4)$ of length $n = 12$.**

Conclusion: **The equidistant permutations exist for and only for $n \in \{4, 12\}$ and $n = 4k+2$ with $k \geq 0$** (including the trivial case $n = 2$).

Further thoughts: What happens if we plot the permutation in three-dimensional space, using triples of consecutive terms of the permutation, rather than their pairs? Some of the observations can be generalized easily – instead of having two numbers, A and B, we'd have three of them and their greatest common divisor would still have to be equal to 1 (althought they wouldn't need to be mutually coprime!). However, the idea of "forced" neighbours breaks – since each term of the permutation only makes use of two of its three neighbour-distances.

At the first glance, it seems that it should be easier to satisfy the 3-dimensional conditions than the two-dimensional ones; after all, there are three parameters rather than two. However, a quick check suggests that this is not the case – apart from the trivial case of $n = 3$, the only values of n corresponding to equidistant permutations seem to be of the form $n = 12k + 6$. There are four distinct permutations (ignoring shifts and reversal) for $n = 6$: $(1, 4, 5, 6, 3, 2)$, $(1, 3, 5, 6, 4, 2)$, $(1, 5, 4, 6, 2, 3)$ and $(1, 5, 3, 6, 2, 4)$.

For greater values of $n = 12k + 6$, the situation becomes more systematic – the corresponding permutations can be built from each other iteratively by starting with $(6, 4, 5, \spadesuit, 1, 3, 2)$ ($\spadesuit$ is just a placeholder, not an actual term of the permutation) and repeating one simple step k times. Step number s (s goes from 1 to k) consists of swapping the two terms following $\spadesuit$, prepending the sequence $(12s - 3, 12s - 1, 12s, 12s + 4, 12s + 6, 12s + 5)$ before $\spadesuit$ and appending $(12s + 1, 12s + 3, 12s + 2, 12s - 2, 12s - 4, 12s - 5)$ after $\spadesuit$.

It's easy to see two simple invariants maintained by the steps: The absolute differences between terms to the left of $\spadesuit$ are $(2, 1, 4)$ and the ones to the right of it are $(4, 2, 1)$ (4 is the difference of

♠'s immediate neighbours). Furthermore, the term to the left of ♠ is equal to 5 modulo 12 and the one to its right is always equal to 1 (modulo 12 as well). These two observations suffice to prove the construction to be correct.
As an example, for $n = 30$, we get the permutation:

$$(6, 4, 5, 9, 11, 12, 16, 18, 17, 21, 23, 24, 28, 30, 29, 25, 27, 26, 22, 20, 19, 15, 13, 14, 10, 8, 7, 3, 1, 2)$$

For now, we don't know if the permutations of this form (and the three extras for $n = 6$) are the only 3-dimensional equidistant permutations.

Even more speculations: Why stop with three-dimensional world? What mysteries await us in four dimensions? Clearly, all 2-dimensional equidistant permutations of length greater than 4 are also 4-dimensionally equidistant. For $n = 4k + 2$, it should be easy to see that these are the only possibilities. However, unlike the 2-dimensional case, $n = 4k$ seems to be pretty abundant with four-dimensional equidistant permutations. Perhaps they exist for all reasonable[4] choices of n?

That's all folks![5]

[4] Surely, n cannot be smaller than 4 and cannot be odd.
[5] Sorry, no wild guesses about even-more-dimensional spaces yet. :-)