On The Mysterious B Sequence

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First, we'll prove a lemma about the solution of the recurrence given in problem statement.

Lemma: The sequence defined recurrently as

$$\begin{aligned} b_0 &= c \\ b_k &= (c+2kd) - \frac{(c+(2k-1)d)^2}{4b_{k-1}} \quad (k>0) \end{aligned}$$

can be expressed explicitly (as long as the sum (2) is well-defined) as

$$b_k = \frac{1}{2}\left(c + (2k+1)d + \frac{1}{S(k)}\right) \tag{1}$$

where

$$S(k) = \sum_{t=0}^{k} \frac{1}{c+(2t-1)d} \tag{2}$$

This statement can be proved by induction fairly easily – but the proof looks a bit long with all the c's and d's. Thus, we'll generalize the statement first, while making the proof more readable at the same time.

Generalized lemma: For any sequence $P = (p_0, p_1, p_2, \ldots)$ of real numbers, the recurrence

$$\begin{aligned} b_0 &= \frac{1}{2}(p_0 + p_1) \\ b_k &= \frac{1}{2}(p_k + p_{k+1}) - \frac{p_k^2}{4b_{k-1}} \end{aligned}$$

can be solved explicitly (as long as the sum (4) is well-defined) as

$$b_k = \frac{1}{2}\left(p_{k+1} + \frac{1}{S(k)}\right) \tag{3}$$

where

$$S(k) = \sum_{t=0}^{k} \frac{1}{p_t} \tag{4}$$

It's easy to see that the original lemma corresponds to $p_k = c + (2k-1)d$.

Proof (by induction)

Base case:

$$b_0 = \frac{1}{2}\left(p_1 + \frac{1}{\frac{1}{p_0}}\right) = \frac{1}{2}(p_0 + p_1)$$

Inductive case: Plugging $b_{k-1} = \frac{1}{2}\left(p_k + \frac{1}{S(k-1)}\right)$ into the recurrence yields

$$
\begin{aligned}
b_k &= \frac{1}{2}(p_k + p_{k+1}) - \frac{p_k^2}{2\left(p_k + \frac{1}{S(k-1)}\right)} \\
&= \frac{1}{2}\left(p_{k+1} + p_k - \frac{p_k^2}{p_k + \frac{1}{S(k-1)}}\right) \\
&= \frac{1}{2}\left(p_{k+1} + \frac{p_k\left(p_k + \frac{1}{S(k-1)}\right) - p_k^2}{p_k + \frac{1}{S(k-1)}}\right) \\
&= \frac{1}{2}\left(p_{k+1} + \frac{\frac{p_k}{S(k-1)}}{p_k + \frac{1}{S(k-1)}}\right) \\
&= \frac{1}{2}\left(p_{k+1} + \frac{1}{S(k-1) + \frac{1}{p_k}}\right) \\
&= \frac{1}{2}\left(p_{k+1} + \frac{1}{S(k)}\right)
\end{aligned}
$$

This completes the proof of the generalized lemma.

Now, we're ready to answer the original question – which conditions on c and d guarantee $b_k > 0$ for all $k \geq 0$?

($c > d$) Each term in (2) is positive and thus so is the whole b_k.

($c = d$) Then, $b_k = (k+1)c$ (as can be easily verified), so it is always positive.

($c < d$) Then, we have $S(0) < 0$, but $S(k) \longrightarrow +\infty$ because it is just the general harmonic series. Thus, there is some index k such that $S(k) < 0$ but $S(k+1) > 0$. Since $S(k+1) = S(k) + \frac{1}{c+(2k+1)d}$, we have

$$0 > S(k) > -\frac{1}{c + (2k+1)d}$$

or, put differently,

$$\frac{1}{S(k)} < -(c + (2k+1)d)$$

Moving right-hand side of the inequality to the left yields $b_k < 0$.

Putting it all together, $b_k > 0$ for all $k \geq 0$ if and only if $c \geq d > 0$. This resolves the original problem.

Note that thanks to the generalized lemma, the same kind of reasoning could be applied to many other sequences, of non-arithmetical kind. For example, if $p_k > 0$ for all $k \geq 0$, the resulting sequence b_k will be positive too. If we set $p_0 < 0$ and the rest of p_k to any positive series whose sum of reciprocals diverges (for example, $p_0 = -1$, $p_k = k$-th prime number), the sequence b_k will contain a negative term.